

THE SEMIDIRECTLY CLOSED PSEUDOVARIETY GENERATED BY APERIODIC BRANDT SEMIGROUPS

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ABSTRACT. This paper presents a study of the semidirectly closed pseudovariety generated by the aperiodic Brandt semigroup B_2 , denoted $\mathbf{V}^*(B_2)$. We construct a basis of pseudoidentities for the semidirect powers of the pseudovariety generated by B_2 which leads to the main result, which states that $\mathbf{V}^*(B_2)$ is decidable.

Independently, using some suggestions given by J. Almeida in his book “Finite Semigroups and Universal Algebra”, we constructed an algorithm to solve the membership problem in $\mathbf{V}^*(B_2)$.

1. INTRODUCTION

Recall that a pseudovariety of semigroups is a class of finite semigroups closed under taking divisors and finite direct products. The semidirect product $\mathbf{V} * \mathbf{W}$ of two pseudovarieties of semigroups is the pseudovariety generated by all semidirect products of semigroups of \mathbf{V} by semigroups of \mathbf{W} [7]. This definition gives an operation on the set of pseudovarieties that is associative and whose idempotents are precisely the semidirectly closed (abbreviated s.c.) pseudovarieties. In the theory of finite semigroups, the semidirect product is a construction that has drawn the attention of many authors. The study of the decidability of semidirect product pseudovarieties assumes particular relevance, since decidability of pseudovarieties is not preserved by the semidirect product operation [11].

The pseudovariety $\mathbf{V}^*(B_2)$ is the smallest s.c. pseudovariety that contains B_2 , where B_2 is the aperiodic Brandt semigroup with five elements. In the lattice of s.c. pseudovarieties of aperiodic semigroups, $\mathbf{V}^*(B_2)$ covers \mathbf{R} (the pseudovariety of all semigroups whose Green

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relation \mathcal{R} is trivial) and $\mathbf{V}^*(B_2)$ is a subpseudovariety of $\mathbf{ER} \cap \mathbf{LR}$ (the intersection of the pseudovariety of all semigroups whose subsemigroups generated by idempotents are members of \mathbf{R} with the pseudovariety of all semigroups whose local subsemigroups are in \mathbf{R}) [1]. In [13] has been proved that $\mathbf{V}^*(B_2) \neq \mathbf{ER} \cap \mathbf{LR}$ and that the interval $[\mathbf{V}^*(B_2), \mathbf{ER} \cap \mathbf{LR}]$ contains a chain, for the inclusion relation, of s.c. pseudovarieties isomorphic to the chain of real numbers.

In this work we are going to study $\mathbf{V}^*(B_2)$ in order to prove its decidability. The pseudovariety $\mathbf{V}^*(B_2)$ is the direct union of the semidirect powers of $\mathbf{V}(B_2)$ (the pseudovariety generated by B_2) whose bases of pseudoidentities are constructed, recursively, using theorems [6, theorem 5.3] and [4, theorem 5.9]. Such bases are not effective, but they are used to obtain a decision criterion for the membership problem for $\mathbf{V}^*(B_2)$.

In [1], J. Almeida gives a suggestion to study the decidability of the membership problem of $\mathbf{V}^*(B_2)$. This arguments and the knowledge of the validity of some pseudoidentities in $\mathbf{V}^*(B_2)$ lead to an independent proof of decidability of $\mathbf{V}^*(B_2)$, because they allow us to construct an algorithm of polynomial complexity to solve the membership problem for $\mathbf{V}^*(B_2)$.

The basic results that we are going to use, about pseudovarieties of semigroups and of semigroupoids that contain B_2 , are proved in [4, 13] and are based on the theory of finite semigroupoids and some previous results obtained by B. Tilson in [14] and by N. Reilly in [9].

2. PRELIMINARIES

For general background and terminology from the classical theory of semigroups, profinite semigroups and semidirect products, the reader is referred to [8, 1, 5, 14, 6].

In this paper we are not going to consider empty algebras. We are going to consider topological algebras (total or partial) and, in particular, finite algebras are supposed to be discrete topological spaces.

In this section we are going to present a brief description of the most relevant basic concepts and propositions in order to make it easier to read the following sections.

2.1. Free profinite semigroups.

For a set X endowed with a topology, we say that a semigroup S is X -generated if there is a continuous function, from X to S , such that S is the smallest closed semigroup that contains the image of X .

For a class V of finite semigroups, we say that S is pro- V if it is the projective limit of semigroups of V .

For a pseudovariety \mathbf{V} and a set X , the X -generated elements of \mathbf{V} form a directed system and the respective projective limit will be denoted $\overline{\Omega}_X \mathbf{V}$. The semigroup $\overline{\Omega}_X \mathbf{V}$ is the free semigroup over X for the class of all pro- \mathbf{V} semigroups, which means that, for every pro- \mathbf{V} semigroup S and continuous function $\varphi : X \rightarrow S$, there is a unique continuous morphism $\overline{\varphi} : \overline{\Omega}_X \mathbf{V} \rightarrow S$ extending φ . The free semigroup over X for the variety generated by \mathbf{V} is denoted $\Omega_X \mathbf{V}$ and is a dense subsemigroup of $\overline{\Omega}_X \mathbf{V}$.

The elements of $\overline{\Omega}_X \mathbf{V}$ are usually called implicit operations, since $\overline{\Omega}_X \mathbf{V}$ is isomorphic to the semigroup of implicit operations on \mathbf{V} over X . If X is an alphabet that contains x , then the implicit operation represented by x^ω is such that, for each finite semigroup S and each continuous function $\varphi : X \rightarrow S$, if $\varphi(x) = s$ then $\overline{\varphi}(x^\omega) = s^n$, the power of s that is idempotent.

Given a profinite set X and a pseudovariety \mathbf{V} (usually $\mathbf{V} = \mathbf{S}$, the pseudovariety of all finite semigroups), a pseudoidentity over \mathbf{V} is a formal equality $\pi = \rho$ where $\pi, \rho \in \overline{\Omega}_X \mathbf{V}$. A semigroup $S \in \mathbf{V}$ satisfies the pseudoidentity $\pi = \rho$, or equivalently, $\pi = \rho$ holds in S , if for every continuous function $\varphi : X \rightarrow S$, $\overline{\varphi}(\pi) = \overline{\varphi}(\rho)$ and we write $S \models \pi = \rho$. For a subclass C of \mathbf{V} we say that C satisfies the pseudoidentity $\pi = \rho$ if all elements of C satisfy $\pi = \rho$, and we write $C \models \pi = \rho$. For a set Σ of pseudoidentities, $[\Sigma]$ denotes the class of all finite semigroups in which all pseudoidentities of Σ hold. In [10] J. Reiterman proves that the set of pseudovarieties and the set of classes of the form $[\Sigma]$ are equal, where Σ is a set of pseudoidentities over finite sets. If $\mathbf{V} = [\Sigma]$ then Σ is said to be a basis of pseudoidentities of \mathbf{V} .

Given $\pi \in \overline{\Omega}_X \mathbf{S}$, $\rho \in \overline{\Omega}_X \mathbf{S}$ is a factor of π if there are $\pi_1, \pi_2 \in (\overline{\Omega}_X \mathbf{S})^1$ such that $\pi = \pi_1 \rho \pi_2$. If $\rho \in X$ and $\pi_1 = 1$ (or $\pi_2 = 1$) then ρ is denoted by $i(\pi)$ (or $t(\pi)$, respectively). By $C(\pi)$ we denote the content of π which is the subset of X consisting of all elements of X that are factors of π .

2.2. Free profinite semigroupoids and semidirect products.

By a (directed) graph G we mean a partial algebra with a support set $V(G) \overset{\circ}{\cup} E(G)$ with two sorts of elements, called vertices and edges respectively, and two binary operations: $\alpha : E(G) \rightarrow V(G)$ and $\omega : E(G) \rightarrow V(G)$. For $c, d \in V(G)$, $G(c, d)$ is the set of edges s of G such that $\alpha(s) = c$ and $\omega(s) = d$. Two edges s_1 and s_2 are said to be consecutive if $\omega(s_1) = \alpha(s_2)$, and are said to be coterminial if

$\alpha(s_1) = \alpha(s_2)$ and $\omega(s_1) = \omega(s_2)$. A finite sequence of consecutive edges of G is a path of G . We define the graph G^+ as the graph such that $V(G^+) = V(G)$, $E(G^+)$ is the set of all paths in G and, for $v_1v_2 \dots v_n$ a path in G , $\alpha(v_1v_2 \dots v_n) = \alpha(v_1)$ and $\omega(v_1v_2 \dots v_n) = \omega(v_n)$. The graph G is said to be connected if, for every pair of vertices (p, q) , there is a finite sequence of vertices v_0, v_1, \dots, v_n such that $v_0 = p$, $v_n = q$ and $G(v_i, v_{i+1}) \cup G(v_{i+1}, v_i) \neq \emptyset$, for $i = 1, \dots, n-1$, and G is said to be strongly connected if, for every pair of vertices (p, q) , $G^+(p, q) \neq \emptyset$.

A semigroupoid S is a graph with an associative partial operation, called composition, whose domain is $\{(s, t) \in E(S)^2 : \alpha(t) = \omega(s)\}$, and such that given (s, t) in the domain their composition is an edge called st which lies in $S(\alpha(s), \omega(t))$. A category S is a semigroupoid that has an identity element at each vertex, which means that, for every $v \in V(S)$, there is $1_v \in S(v, v)$ such that, whenever the compositions are defined in S , $s1_v = s$ and $1_v s = s$. For a semigroupoid S , we denote by S^c the smallest category that contains S . Semigroups can be interpreted as semigroupoids with only one vertex.

For graphs, semigroupoids and categories, morphisms are defined as functions between graphs, semigroupoids and categories, respectively, respecting sorts and operations. A morphism between the semigroupoids S and T , $\psi : S \longrightarrow T$, is said to be:

- faithful if, for every $c, d \in V(S)$, the restriction $\psi|_{S(c, d)}$ is injective;
- a quotient morphism if ψ is surjective and the restriction $\psi|_{V(S)}$ is injective;
- an isomorphism if ψ is bijective.

A semigroupoid S is a quotient of a semigroupoid T if there is a quotient morphism $\psi : T \longrightarrow S$. We say that a semigroupoid S divides a semigroupoid T , and we write $S \prec T$, if S is a quotient of a semigroupoid E for which exists a faithful morphism $\beta : E \longrightarrow T$.

A variety (pseudovariety) of semigroupoids \mathbf{V} is a class of (finite) semigroupoids containing a semigroupoid with just one vertex and one edge which is closed under taking (finite) divisors and (finitary) products and coproducts. For a pseudovariety \mathbf{W} of semigroups we define $g\mathbf{W}$ to be the pseudovariety of semigroupoids generated by the elements of \mathbf{W} .

For semigroupoids we can construct a theory similar to the semigroup case, by defining free profinite semigroupoids. As in the case of semigroups, the projective limit of the G -generated semigroupoids of \mathbf{V} , $\bar{\Omega}_G \mathbf{V}$, is the free semigroupoid generated by the graph G in the class of all pro- \mathbf{V} semigroupoids. The concept of implicit operation of semigroupoids is obtained by extension of the concept of implicit operation

of semigroups, in such a way that the set of all implicit operations assumes the structure of a semigroupoid that is isomorphic to $\overline{\Omega}_G \mathbf{V}$.

A pseudoidentity is a formal equality between two coterminal edges π and ρ of $\overline{\Omega}_G \mathbf{V}$, denoted $(\pi = \rho, G)$. In case G is finite and connected, we say that a semigroupoid $S \in \mathbf{V}$ satisfies a pseudoidentity $(\pi = \rho, G)$ (or that $(\pi = \rho, G)$ holds in S) if $\overline{\varphi}(\pi) = \overline{\varphi}(\rho)$, for every graph morphism $\varphi : G \longrightarrow S$ and $\overline{\varphi} : \overline{\Omega}_G \mathbf{V} \longrightarrow S$ the unique continuous extension of φ . Usually we will consider pseudoidentities on \mathbf{Sd} (the pseudovariety of all finite semigroupoids). An extension of Reiterman's Theorem is valid, which means that each pseudovariety of semigroupoids can be defined by a set of pseudoidentities over finite connected graphs.

By a labeling of a graph G by $\overline{\Omega}_X \mathbf{S}$ we mean a pair $((\pi_q)_{q \in V(G)}, \varepsilon)$ where $(\pi_q)_{q \in V(G)}$ is a family of elements of $(\overline{\Omega}_X \mathbf{S})^1$ and $\varepsilon : E(G) \longrightarrow \overline{\Omega}_X \mathbf{S}$ is a map. Such labeling is compatible with a pseudovariety \mathbf{V} if, for all $s \in E(G)$, $\mathbf{V} \models (\pi_{\alpha(s)} \cdot \varepsilon(s) = \pi_{\omega(s)})$. The process to construct a basis of pseudoidentities for pseudovarieties of semigroups of the form $\mathbf{V} * \mathbf{W}$ is stated in the following theorem.

Theorem 2.1. [6, theorem 5.3] *Let \mathbf{V} and \mathbf{W} be pseudovarieties of semigroups. If Σ is a pseudoidentity basis of $g\mathbf{V}$, then the set of the pseudoidentities of the form*

$$\pi_{\alpha(u)} \varepsilon(u) = \pi_{\alpha(v)} \varepsilon(v),$$

*where $(u = v, G) \in \Sigma$, X is a finite set, $((\pi_q)_{q \in V(G)}, \varepsilon')$ is a labeling of G by $\overline{\Omega}_X \mathbf{S}$ compatible with \mathbf{W} and $\varepsilon : \overline{\Omega}_G \mathbf{Sd} \longrightarrow \overline{\Omega}_X \mathbf{S}$ is the continuous morphism of semigroupoids constant on vertices and defined on edges by: $\varepsilon(s) = \varepsilon'(s)$, for every $s \in E(G)$, is a pseudoidentity basis for the pseudovariety of semigroups $\mathbf{V} * \mathbf{W}$. \square*

2.3. Some results about pseudovarieties which contain B_2 .

By definition, a Brandt semigroup is a completely 0-simple inverse semigroup. So a Brandt semigroup is isomorphic to a $I \times I$ Rees matrix semigroup over a 0-group, G^0 , with the identity matrix. Usually we denote by B_n the aperiodic Brandt semigroup such that $\#I = n$. The set $\{x^2 = x^3, x^2y^2 = y^2x^2, x(yx)^2 = xyx\}$ is an identity basis of the variety of semigroups generated by B_2 , $V(B_2)$, [15]. The pseudovariety of semigroups generated by B_2 , $\mathbf{V}(B_2)$, is equal to $V(B_2) \cap \mathbf{S}$ and so

$$\mathbf{V}(B_2) = \llbracket x^2 = x^3, x^2y^2 = y^2x^2, x(yx)^2 = xyx \rrbracket.$$

A generalization of N. Reilly's solution of the word problem in free inverse semigroups to the profinite semigroups $\overline{\Omega}_X \mathbf{V}(B_2)$ was obtained in [4] and the result is presented in theorem 2.2 below. To understand the meaning of this theorem we need to introduce some notions and terminology.

For a set X , $X^{-1} = \{x^{-1} : x \in X\}$ is a disjoint copy of X . Given a finite alphabet X and $\pi \in \overline{\Omega}_X \mathbf{S}$, δ_π is the equivalence relation over $C(\pi) \cup C(\pi)^{-1}$ generated by the set

$$\{(x^{-1}, y) : x, y \in C(\pi), xy \text{ is a factor of } \pi\}.$$

Given an alphabet X and $\pi \in \overline{\Omega}_X \mathbf{S}$, we define \mathcal{A}_π to be the graph:

- $V(\mathcal{A}_\pi) = (C(\pi) \cup C(\pi)^{-1})/\delta_\pi$;
- for any $v_1, v_2 \in V(\mathcal{A}_\pi)$,

$$\mathcal{A}_\pi(v_1, v_2) = \{x \in C(\pi) : [x]_{\delta_\pi} = v_1 \text{ and } [x^{-1}]_{\delta_\pi} = v_2\}.$$

The vertices $[i(\pi)]_{\delta_\pi}$ and $[t(\pi)^{-1}]_{\delta_\pi}$ are called the initial vertex and the final vertex of \mathcal{A}_π , respectively.

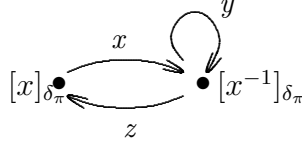
Since the $\mathbf{V}(B_2)$ is finitely generated, $\Omega_X \mathbf{V}(B_2) = \overline{\Omega}_X \mathbf{V}(B_2)$ and, using [9, theorem 3.3], the proof of the next theorem is an obvious consequence of the fact that the function that associates to each implicit operation π the relation δ_π , is a continuous function.

Theorem 2.2. [4, theorem 5.6] *Given a finite alphabet X and $\pi_1, \pi_2 \in \overline{\Omega}_X \mathbf{S}$, the pseudoidentity $\pi_1 = \pi_2$ holds in B_2 if and only if:*

1. $\mathcal{A}_{\pi_1} = \mathcal{A}_{\pi_2}$;
2. $[i(\pi_1)]_{\delta_{\pi_1}} = [i(\pi_2)]_{\delta_{\pi_2}}$ and $[t(\pi_1)^{-1}]_{\delta_{\pi_1}} = [t(\pi_2)^{-1}]_{\delta_{\pi_2}}$. □

It follows from the definition that, if u is a word factor of π then u defines a path in \mathcal{A}_π from $[i(u)]_{\delta_\pi}$ to $[t(u)^{-1}]_{\delta_\pi}$.

In [3] it was proved that $E(\overline{\Omega}_G \mathbf{Sd})$ can be identified with a subset of $\overline{\Omega}_{E(G)} \mathbf{S}$ (proposition 2.3). Lemma 5.7 in [4] states an opposite relation, in the sense that, for any finite set X and for each $\pi \in \overline{\Omega}_X \mathbf{S}$ we can associate an edge implicit operation in $\overline{\Omega}_{\mathcal{A}_\pi} \mathbf{Sd}$, again denoted by π . Note that, if $(w_n)_n$ is a sequence of words in X^+ converging to π , then there is an integer p such that, for all $n \geq p$, w_n labels a path in \mathcal{A}_π from the initial vertex to the final one and all edges of \mathcal{A}_π occur in w_n . For example, let $\pi = xy^\omega zx$ and consider the sequence $(xy^{n!}zx)_n$ that converges to π . The graph \mathcal{A}_π is



and, for every n , $xy^{n!}zx$ denotes a path from $[x]_{\delta_\pi}$ to $[x^{-1}]_{\delta_\pi}$. The limit of the sequence of paths $(xy^{n!}zx)_n$ is the edge implicit operation denoted by $xy^\omega zx$.

Theorem 2.3. [4, theorem 5.9] *Let \mathbf{V} be a pseudovariety of semigroups such that $B_2 \in \mathbf{V}$. If $\mathbf{V} = \llbracket u_i = v_i : i \in I \rrbracket$ then $g\mathbf{V} = \llbracket (u_i = v_i, \mathcal{A}_{u_i}) : i \in I \rrbracket$.* \square

Corollary 2.4. *The pseudovariety $g\mathbf{V}(B_2)$ is the pseudovariety*

$$\llbracket (x^3 = x^2, \mathcal{A}_{x^2}), (x^2y^2 = y^2x^2, \mathcal{A}_{x^2y^2}), (x(yx)^2 = xyx, \mathcal{A}_{xyx}) \rrbracket.$$

\square

3. THE SEMIDIRECT POWERS OF $\mathbf{V}(B_2)$

The semidirect powers of a pseudovariety of semigroups \mathbf{W} are:

- $\mathbf{W}^0 = \llbracket x = y \rrbracket$ the pseudovariety of trivial semigroups;
- $\mathbf{W}^n = \mathbf{W} * \mathbf{W}^{n-1} (= \mathbf{W}^{n-1} * \mathbf{W})$, for every $n \geq 1$.

So, $\mathbf{W}^1 = \mathbf{W}$, $\mathbf{W}^{n-1} \subseteq \mathbf{W}^n$, the family $(\mathbf{W}^n)_{n \geq 0}$ is a chain and $\bigcup_{n \geq 0} \mathbf{W}^n$ is the s.c. pseudovariety generated by \mathbf{W} , usually represented by \mathbf{W}^* . The computation of the semidirect powers of $\mathbf{V}(B_2)$ can be done recursively using in each step theorems 2.1 and 2.3.

Proposition 3.1. *Let G be a graph with a vertex v such that, for every $v' \in V(G) \setminus \{v\}$, $G^+(v, v') \neq \emptyset$. A labeling*

$$((\pi_q)_{q \in V(G)}, \varepsilon' : E(G) \longrightarrow \overline{\Omega}_X \mathbf{S})$$

is compatible with a pseudovariety \mathbf{V} if and only if, for every $q \in V(G)$ and $w \in G^+(v, q)$, $\mathbf{V} \models \pi_v \varepsilon(w) = \pi_q$, where $\varepsilon : \overline{\Omega}_G \mathbf{Sd} \longrightarrow \overline{\Omega}_X \mathbf{S}$ is the continuous morphism of semigroupoids constant on vertices and defined on edges by: $\varepsilon(s) = \varepsilon'(s)$, for every $s \in E(G)$.

Proof

The condition is obviously necessary. In order to prove the inverse implication, let $s \in E(G)$ and $w_{\alpha(s)} \in G^+(v, \alpha(s))$. By hypothesis

$$\mathbf{V} \models \pi_v \varepsilon(w_{\alpha(s)}) = \pi_{\alpha(s)} \text{ and } \mathbf{V} \models \pi_v \varepsilon(w_{\alpha(s)} s) = \pi_{\omega(s)}.$$

Hence, $\mathbf{V} \models \pi_{\alpha(s)} \varepsilon(s) = \pi_{\omega(s)}$. \square

Corollary 3.2. *Let G be a graph in the conditions of proposition 3.1, $\pi \in \overline{\Omega}_X \mathbf{S}$ and $\varepsilon' : E(G) \rightarrow \overline{\Omega}_X \mathbf{S}$ such that, for every $q \in V(G)$ and $w_1, w_2 \in G^+(v, q)$, $\mathbf{V} \models \pi \varepsilon(w_1) = \pi \varepsilon(w_2)$ and, in case $v = q$, $\mathbf{V} \models \pi \varepsilon(w_1) = \pi$, where $\varepsilon : \overline{\Omega}_G \mathbf{Sd} \rightarrow \overline{\Omega}_X \mathbf{S}$ is the continuous morphism of semigroupoids constant on vertices and defined on edges by: $\varepsilon(s) = \varepsilon'(s)$, for every $s \in E(G)$. Then, the set of the labelings of the form $((\pi_q)_{q \in V(G)}, \varepsilon')$ such that $\pi_v = \pi$ which are compatible with \mathbf{V} is nonempty.*

Proof

Let $\varphi : \overline{\Omega}_X \mathbf{S} \rightarrow \overline{\Omega}_X \mathbf{V}$ be the projection morphism. If we choose $\pi_v = \pi$ and, for each $q \in V(G) \setminus \{v\}$ and some $w_1 \in G^+(v, q)$, $\pi_q \in \varphi^{-1}(\varphi(\pi \varepsilon(w_1)))$, then we obtain a family $(\pi_q)_{q \in V(G)}$ such that $((\pi_q)_{q \in V(G)}, \varepsilon')$ define a labeling of G . By proposition 3.1, such labeling is compatible with \mathbf{V} . Making all possible choices we obtain the set of labelings referred to in the statement. \square

In the conditions of the corollary 3.2 we will say that π and ε define a set of labelings of G by $\overline{\Omega}_X \mathbf{S}$ compatible with \mathbf{V} or, more precisely, π and ε define the set of labelings referred to in the proof.

This result applied to the calculation of the pseudovariety $\mathbf{V}^2(B_2)$, using theorem 2.1 and the pseudoidentity basis Γ of $g\mathbf{V}(B_2)$ calculated in corollary 2.4, leads to the conclusion that

$$\begin{aligned} \Sigma = \{ \pi \varepsilon(u) = \pi \varepsilon(v) : X \text{ is a finite set, } \pi \in (\overline{\Omega}_X \mathbf{S})^1 \\ \text{and } \varepsilon : \overline{\Omega}_{\mathcal{A}_u} \mathbf{Sd} \rightarrow \overline{\Omega}_X \mathbf{S} \text{ define a set of labelings} \\ \text{compatible with } \mathbf{V}(B_2), \text{ and } (u = v, \mathcal{A}_u) \in \Gamma \} \end{aligned}$$

is a pseudoidentity basis of $\mathbf{V}^2(B_2)$.

Note that for $(u = v, \mathcal{A}_u) \in \Sigma$ and all labelings member of the set of labelings defined by fixed π and ε corresponds only one pseudoidentity on the basis of $\mathbf{V}^2(B_2)$. The verification that a labeling is compatible with $\mathbf{V}(B_2)$ is based on theorem 2.2.

In order to give a pseudoidentity basis of pseudovarieties of the form $\mathbf{V}^n(B_2)$ we need to prove the following lemmas.

Lemma 3.3. *Let $\pi \in \overline{\Omega}_X \mathbf{S}$ and ρ be a factor of π . Then there is a graph morphism $\varphi_{\rho, \pi} : \mathcal{A}_\rho \rightarrow \mathcal{A}_\pi$ such that $\varphi_{\rho, \pi}(\mathcal{A}_\rho)$ is the support graph of $C(\rho)$.*

Proof

Recall that $C(\rho) \subseteq C(\pi) = E(\mathcal{A}_\pi)$. Consider the graph morphism

$$\begin{array}{ccc} \varphi_{\rho,\pi} : & \mathcal{A}_\rho & \longrightarrow \mathcal{A}_\pi \\ & [x]_{\delta_\rho} & \longmapsto [x]_{\delta_\pi} \\ & x & \longmapsto x . \end{array}$$

Since $\delta_\rho \subseteq \delta_\pi$, $\varphi_{\rho,\pi}$ is a graph morphism injective on edges. \square

Lemma 3.4. *Let $\pi, u \in \overline{\Omega}_X \mathbf{S}$ and $\varepsilon : \overline{\Omega}_{\mathcal{A}_u} \mathbf{Sd} \longrightarrow \overline{\Omega}_X \mathbf{S}$ be such that π and ε define a set of labelings of \mathcal{A}_u compatible with $\mathbf{V}(B_2)$. If, for some $\pi_a \in \overline{\Omega}_X \mathbf{S}$, $\pi_b \in (\overline{\Omega}_X \mathbf{S})^1$,*

$$(1) \quad \pi = \pi_a \pi_b \quad \text{and} \quad C(\pi_a) \cap C(\pi_b \varepsilon(u)) = \emptyset,$$

then π_b and ε define a set of labelings of \mathcal{A}_u compatible with $\mathbf{V}(B_2)$.

Proof

Let v be the initial vertex of \mathcal{A}_u , $q \in V(\mathcal{A}_u)$ and $w, w' \in \mathcal{A}_u(v, q)$. Since $\pi = \pi_a \pi_b$, the graph $\mathcal{A}_{\pi \varepsilon(w)}$ is the union of the subgraphs $\varphi_{\pi_a, \pi \varepsilon(w)}(\mathcal{A}_{\pi_a})$ and $\varphi_{\pi_b \varepsilon(w), \pi \varepsilon(w)}(\mathcal{A}_{\pi_b \varepsilon(w)})$. These two subgraphs have only one common vertex: the image under $\varphi_{\pi_a, \pi \varepsilon(w)}$ of the final vertex of \mathcal{A}_{π_a} is equal to the image under $\varphi_{\pi_b \varepsilon(w), \pi \varepsilon(w)}$ of the initial vertex of $\mathcal{A}_{\pi_b \varepsilon(w)}$. Otherwise, the graph $\mathcal{A}_{\pi \varepsilon(w')}$ is the union of the subgraphs $\varphi_{\pi_a, \pi \varepsilon(w')}(\mathcal{A}_{\pi_a})$ and $\varphi_{\pi_b \varepsilon(w'), \pi \varepsilon(w')}(\mathcal{A}_{\pi_b \varepsilon(w')})$, and these two subgraphs have only one common vertex: the image under $\varphi_{\pi_a, \pi \varepsilon(w')}$ of the final vertex of \mathcal{A}_{π_a} which is the image under $\varphi_{\pi_b \varepsilon(w), \pi \varepsilon(w)}$ of the initial vertex of $\mathcal{A}_{\pi_b \varepsilon(w)}$. Moreover, by condition (1), the graph morphisms $\varphi_{\pi_a, \pi \varepsilon(w)}$, $\varphi_{\pi_a, \pi \varepsilon(w')}$, $\varphi_{\pi_b \varepsilon(w), \pi \varepsilon(w)}$ and $\varphi_{\pi_b \varepsilon(w'), \pi \varepsilon(w')}$ are injective on edges and on vertices.

By hypothesis π and ε define a set of labelings of \mathcal{A}_u compatible with $\mathbf{V}(B_2)$, so $\mathcal{A}_{\pi \varepsilon(w)} = \mathcal{A}_{\pi \varepsilon(w')}$ and the final vertices are the same which means that $[t(\pi \varepsilon(w))]_{\delta_{\pi \varepsilon(w)}}^{-1} = [t(\pi \varepsilon(w'))]_{\delta_{\pi \varepsilon(w')}}^{-1}$. Consequently, $\varphi_{\pi_b \varepsilon(w), \pi \varepsilon(w)}(\mathcal{A}_{\pi_b \varepsilon(w)}) = \varphi_{\pi_b \varepsilon(w'), \pi \varepsilon(w')}(\mathcal{A}_{\pi_b \varepsilon(w')})$, which implies $\mathcal{A}_{\pi_b \varepsilon(w)} = \mathcal{A}_{\pi_b \varepsilon(w')}$, and the initial and the final vertices of $\mathcal{A}_{\pi_b \varepsilon(w)}$ and $\mathcal{A}_{\pi_b \varepsilon(w')}$ are the same, respectively, because

$$\begin{aligned} [i(\pi_b \varepsilon(w))]_{\delta_{\pi_b \varepsilon(w)}} &= \varphi_{\pi_b \varepsilon(w), \pi \varepsilon(w)}^{-1}([t(\pi_a)^{-1}]_{\delta_{\pi \varepsilon(w)}}) = \\ &= \varphi_{\pi_b \varepsilon(w'), \pi \varepsilon(w')}^{-1}([t(\pi_a)^{-1}]_{\delta_{\pi \varepsilon(w')}}) = [i(\pi_b \varepsilon(w'))]_{\delta_{\pi_b \varepsilon(w')}} \end{aligned}$$

and

$$\begin{aligned} [t(\pi_b \varepsilon(w))]_{\delta_{\pi_b \varepsilon(w)}}^{-1} &= \varphi_{\pi_b \varepsilon(w), \pi \varepsilon(w)}^{-1}([t(\pi \varepsilon(w))]_{\delta_{\pi \varepsilon(w)}}^{-1}) = \\ &= \varphi_{\pi_b \varepsilon(w'), \pi \varepsilon(w')}^{-1}([t(\pi \varepsilon(w'))]_{\delta_{\pi \varepsilon(w')}}^{-1}) = [t(\pi_b \varepsilon(w'))]_{\delta_{\pi_b \varepsilon(w')}}^{-1} \end{aligned}$$

□

For $\pi, \pi_a, u \in \overline{\Omega}_X \mathbf{S}$, $\pi_b \in \overline{\Omega}_X \mathbf{S}^1$ and $\varepsilon : \overline{\Omega}_{\mathcal{A}_u} \mathbf{Sd} \rightarrow \overline{\Omega}_X \mathbf{S}$ such that $\pi = \pi_a \pi_b$, the pseudoidentity $\pi \varepsilon(u) = \pi \varepsilon(v)$ is a consequence of $\pi_b \varepsilon(u) = \pi_b \varepsilon(v)$. Hence, for example, as a pseudoidentity basis of $\mathbf{V}^2(B_2)$ we can consider the set Σ_2 of the pseudoidentities such as those in Σ and such that the values of π do not satisfy the condition (1).

Lemma 3.5. *Let $\pi \in \overline{\Omega}_X \mathbf{S}$ such that $i(\pi) = t(\pi)$ and $\pi \notin X$. Then \mathcal{A}_π is strongly connected.*

Proof

Note that, for every $u \in \overline{\Omega}_X \mathbf{S}$ and for each $x \in C(u)$, there is exactly one edge of \mathcal{A}_u defined by x , which belongs to a path defined by a word factor of u .

Let $(w_n)_n$ be a sequence of words converging to π . Then there is $p \in \mathbb{N}$ such that, for all $n \geq p$, $i(\pi) = i(w_n) = t(w_n)$, $B_2 \models w_n = \pi$ and, consequently, $(w_n)_{n \geq p}$ is a sequence of paths of \mathcal{A}_π . Moreover, \mathcal{A}_π is the support of each w_n (no proper subgraph of \mathcal{A}_π contains all edges of w_n) and $i(\pi)$ is the initial and final edge of w_n . For $n \geq p$, there is $w'_n \in E(\mathcal{A}_\pi^+)$ such that $w_n = w'_n i(\pi)$ and, consequently, the support graph of w'_n is \mathcal{A}_π and $\alpha(w'_n) = \omega(w'_n) = \alpha(w_n)$ is the initial vertex of \mathcal{A}_π . Hence w'_n is a path of \mathcal{A}_π such that, for each $v \in V(\mathcal{A}_\pi)$, there are factors $u_v \in \mathcal{A}_\pi^+(\alpha(w'_n), v)$ and $u^v \in \mathcal{A}_\pi^+(v, \alpha(w'_n))$. So, for every $v, v' \in V(\mathcal{A}_\pi)$, $u^v u_{v'} \in \mathcal{A}_\pi^+(v, v')$ and $u^{v'} u_v \in \mathcal{A}_\pi^+(v', v)$. □

Note that, if G_1 and G_2 are strongly connected subgraphs of a graph G , such that $V(G) = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) \neq \emptyset$, then G is strongly connected.

Lemma 3.6. *Let X be a finite set and $\pi \in \overline{\Omega}_X \mathbf{S}$ such that $\pi = \pi_1 \pi_2 \pi_3$, where $\pi_1, \pi_2, \pi_3 \in (\overline{\Omega}_X \mathbf{S})^1$, and suppose $\mathcal{A}_{\pi_1 \pi_2}$ and $\mathcal{A}_{\pi_2 \pi_3}$ are strongly connected graphs. Then, \mathcal{A}_π is strongly connected.*

Proof

The graph \mathcal{A}_π is the union of $\varphi_{\pi_1 \pi_2, \pi}(\mathcal{A}_{\pi_1 \pi_2})$ and $\varphi_{\pi_2 \pi_3, \pi}(\mathcal{A}_{\pi_2 \pi_3})$, which are strongly connected graphs and both contain the vertex $[i(\pi_2)]_{\delta_\pi}$, if $\pi_2 \neq 1$, or the vertex $[t(\pi_1)^{-1}]_{\delta_\pi} = [i(\pi_3)]_{\delta_\pi}$, if $\pi_2 = 1$. □

Lemma 3.7. *Let $u \in \overline{\Omega}_X \mathbf{S}$. If $\pi \in \overline{\Omega}_X \mathbf{S}$ and $\varepsilon : \overline{\Omega}_{\mathcal{A}_u} \mathbf{Sd} \rightarrow \overline{\Omega}_X \mathbf{S}$ is a continuous morphism of semigroupoids, which define a set of labelings of \mathcal{A}_u compatible with $\mathbf{V}(B_2)$, then*

- i. *if \mathcal{A}_u is strongly connected then $\mathcal{A}_{\pi \varepsilon(u)} = \mathcal{A}_\pi$;*

- ii. if π does not have a decomposition as in (1) and $C(\varepsilon(u)) \subseteq C(\pi)$, then $\mathcal{A}_{\pi\varepsilon(u)}$ is strongly connected.

Proof

If \mathcal{A}_u is strongly connected then there is $u' \in \mathcal{A}_u^*([t(u)^{-1}]_{\delta_u}, [i(u)]_{\delta_u})$. For any such word u' , $B_2 \models \pi\varepsilon(u)\varepsilon(u') = \pi$ and, consequently, $C(\varepsilon(u)) \subseteq C(\pi)$ and $\delta_\pi \subseteq \delta_{\pi\varepsilon(u)} \subseteq \delta_{\pi\varepsilon(u)\varepsilon(u')} = \delta_\pi$. Hence, $\mathcal{A}_{\pi\varepsilon(u)} = \mathcal{A}_\pi$.

Now, suppose π does not have a decomposition as in (1) and $C(\varepsilon(u)) \subseteq C(\pi)$. If $i(\pi) = t(\varepsilon(u))$ then lemma 3.5 leads to the conclusion that $\mathcal{A}_{\pi\varepsilon(u)}$ is strongly connected. If $i(\pi) \neq t(\varepsilon(u))$ then, since $C(\varepsilon(u)) \subseteq C(\pi)$, there are $\pi_1, \pi_0, u_0 \in (\overline{\Omega}_X \mathbf{S})^1$ such that

$$\pi\varepsilon(u) = l_1\pi_1z\pi_0u_0z$$

where $i(\pi) = l_1$, $t(\varepsilon(u)) = z$ and $u_0z = \varepsilon(u)$, $z \notin C(l_1\pi_1)$ and $C(l_1\pi_1) \cap C(\pi_0u_0) \neq \emptyset$ since π can not be factorized as in (1).

If $l_1 \in C(\pi_0u_0)$ then $\pi\varepsilon(u)$ is of the form $l_1 \cdots z \cdots l_1 \cdots z$ and, by lemmas 3.5 and 3.6, $\mathcal{A}_{\pi\varepsilon(u)}$ is strongly connected. Otherwise, there exist $\pi'_1, \pi_2 \in (\overline{\Omega}_X \mathbf{S})^1$ and $l_2 \in X$ such that $\pi_1 = \pi'_1 l_1 \pi_2$, $l_1 \notin C(\pi_2)$, $l_2 \in C(\pi'_1) \cap C(\pi_2 z \pi_0 u_0 z)$, and

$$\pi\varepsilon(u) = l_1 \pi'_1 l_1 \pi_2 z \pi_0 u_0 z.$$

If $l_2 \in C(\pi_0u_0)$ then $\pi\varepsilon(u)$ is of the form $l_1 \cdots l_2 \cdots l_1 \cdots z \cdots l_2 \cdots z$ and, using lemmas 3.5 and 3.6, we conclude that $\mathcal{A}_{\pi\varepsilon(u)}$ is strongly connected. If $l_2 \notin C(\pi_0u_0)$ then there are $\pi'_2, \pi_3 \in (\overline{\Omega}_X \mathbf{S})^1$ such that $\pi_2 = \pi'_2 l_2 \pi_3$, $l_2 \notin C(\pi_3) \subset C(\pi_2)$, and

$$\pi\varepsilon(u) = l_1 \pi'_1 l_1 \pi'_2 l_2 \pi_3 z \pi_0 u_0 z.$$

Hence, $\mathcal{A}_{l_1 \cdots l_2 \cdots l_1 \pi'_2 l_2}$ is strongly connected and there is $l_3 \in X$ such that $l_3 \in C(\pi'_1 l_1 \pi'_2) \cap C(\pi_3 z \pi_0 u_0 z)$.

Since $C(\pi)$ is finite, after a finite number of steps i , either we conclude that $\mathcal{A}_{\pi\varepsilon(u)}$ is strongly connected or $\pi_{i+1} = 1$. In the second case, $\mathcal{A}_{l_1 \cdots l_{i-1} \pi'_i l_i}$ is strongly connected,

$$\pi\varepsilon(u) = l_1 \pi'_1 l_1 \pi'_2 l_2 \cdots l_{i-1} \pi'_i l_i z \pi_0 u_0 z$$

and, for every $j \in \{1, \dots, i\}$, $l_j \notin C(z\pi_0u_0z)$. Since π can not be factorized as in (1), then, there is $y \in C(l_1 \pi'_1 l_1 \pi'_2 l_2 \cdots l_{i-1} \pi'_i) \cap C(z\pi_0u_0z)$, and $\mathcal{A}_{\pi\varepsilon(u)}$ is the union of $\varphi_{l_1 \cdots \pi'_i l_i, \pi\varepsilon(u)}(\mathcal{A}_{l_1 \cdots \pi'_i l_i})$ and $\varphi_{z\pi_0u_0z, \pi\varepsilon(u)}(\mathcal{A}_{z\pi_0u_0z})$. By lemmas 3.5 and 3.6, we conclude that $\mathcal{A}_{\pi\varepsilon(u)}$ is strongly connected. \square

Applying theorem 2.3 and the above lemmas to the calculation of $g\mathbf{V}^2(B_2)$, we conclude that

$$g\mathbf{V}^2(B_2) = \llbracket (\pi\varepsilon(u) = \pi\varepsilon(v), \mathcal{A}_\pi) : \pi\varepsilon(u) = \pi\varepsilon(v) \in \Sigma_2 \rrbracket$$

where, in each case, $\mathcal{A}_{\pi\varepsilon(u)} = \mathcal{A}_\pi$ is a strongly connected graph.

Theorem 3.8. *For each $n \geq 1$, let Σ_n be the set of all pseudoidentities of the form*

$$\pi_{n-1}\varepsilon_{n-1}(\pi_{n-2} \cdots \varepsilon_2(\pi_1\varepsilon_1(u)) \cdots) = \pi_{n-1}\varepsilon_{n-1}(\pi_{n-2} \cdots \varepsilon_2(\pi_1\varepsilon_1(v)) \cdots),$$

where

- $(u = v) \in \{x^3 = x^2, x^2y^2 = y^2x^2, x(yx)^2 = xyx\}$,
- for $i = 1, \dots, n-1$, X_i is a finite set, $\pi_i \in \overline{\Omega}_{X_i}\mathbf{S}$ and $\varepsilon_i : \overline{\Omega}_{\mathcal{A}_{\pi_{i-1}}}\mathbf{Sd} \longrightarrow \overline{\Omega}_{X_i}\mathbf{S}$ define a set of labelings of $\mathcal{A}_{\pi_{i-1}}$ compatible with $\mathbf{V}(B_2)$, such that each π_i can not be factorized as in (1) relatively to $\varepsilon_i(\pi_{i-1} \cdots \varepsilon_1(u) \cdots)$ (by π_0 we mean u).

Then Σ_n is a pseudoidentity basis of $\mathbf{V}^n(B_2)$, and

$$\mathcal{A}_{\pi_{n-1}} = \mathcal{A}_{\pi_{n-1}\varepsilon_{n-1}(\pi_{n-2} \cdots \varepsilon_2(\pi_1\varepsilon_1(u)) \cdots)}$$

is strongly connected.

Proof

The proof is by induction on n . The statement is valid for $n = 1, 2$, as we have noted before.

By induction hypothesis suppose that, for some $p \geq 1$, the statement is valid. So, if Γ_p is the set

$$\begin{aligned} & \{(\pi_{p-1}\varepsilon_{p-1}(\pi_{p-2} \cdots \varepsilon_1(u) \cdots) = \pi_{p-1}\varepsilon_{p-1}(\pi_{p-2} \cdots \varepsilon_1(v) \cdots), \mathcal{A}_{\pi_{p-1}}) : \\ & \pi_{p-1}\varepsilon_{p-1}(\pi_{p-2} \cdots \varepsilon_1(u) \cdots) = \pi_{p-1}\varepsilon_{p-1}(\pi_{p-2} \cdots \varepsilon_1(v) \cdots) \in \Sigma_p\}, \end{aligned}$$

then $g\mathbf{V}^p(B_2) = \llbracket \Gamma_p \rrbracket$ and, by lemma 3.7, the graph $\mathcal{A}_{\pi_{p-1}}$ is equal to the strongly connected graph $\mathcal{A}_{\pi_{p-1}\varepsilon_{p-1}(\pi_{p-2} \cdots \varepsilon_1(u) \cdots)}$. Hence, by theorem 2.1, we conclude that the pseudovariety $\mathbf{V}^{p+1}(B_2)$ is defined by the pseudoidentities of the form

$$\pi_p\varepsilon_p(u_p) = \pi_p\varepsilon_p(v_p)$$

where

- $(u_p = v_p, \mathcal{A}_{\pi_{p-1}}) \in \Gamma_p$,
- X_p a finite set, $\pi_p \in \overline{\Omega}_{X_p}\mathbf{S}$ and $\varepsilon_p : \overline{\Omega}_{\mathcal{A}_{\pi_{p-1}}}\mathbf{Sd} \longrightarrow \overline{\Omega}_{X_p}\mathbf{S}$ define a set of labelings of $\mathcal{A}_{\pi_{p-1}}$ compatible with $\mathbf{V}(B_2)$ and, by lemma 3.4, π_p can not be factorized as in (1) relatively to $\varepsilon_p(u_p)$.

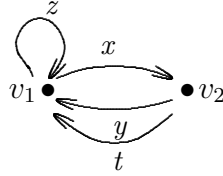
The graph $\mathcal{A}_{\pi_{p-1}} (= \mathcal{A}_{u_p})$ is strongly connected, by induction hypothesis, and then, by lemma 3.7, \mathcal{A}_{π_p} is equal to $\mathcal{A}_{\pi_p \varepsilon_p(u_p)}$ which is strongly connected. \square

Corollary 3.9. *The pseudovariety of semigroupoids $g\mathbf{V}^n(B_2)$ is defined by pseudoidentities over strongly connected graphs, for $n \geq 1$. \square*

Using the same arguments, we can prove that some pseudoidentities hold in $\mathbf{V}^*(B_2)$. As an example, consider an alphabet X such that $\{x, y, z, t\} \subseteq X$ and the pseudoidentity

$$(zxyxtz)^\omega x(yx)^2 = (zxyxtz)^\omega xyx$$

which holds in $\mathbf{V}(B_2)$, by theorem 2.2. The graph $\mathcal{A}_{(zxyxtz)^\omega xyx}$ is presented in the next picture, where $v_1 = \{z^{-1}, z, x, t^{-1}, y^{-1}\}$ is the initial vertex and $v_2 = \{x^{-1}, t, y\}$ is the final vertex.



Suppose that the pseudoidentity holds in $\mathbf{V}^i(B_2)$, for some $i \geq 1$. The labeling defined by $\pi = (zxyxtz)^\omega$ and $\varepsilon : \overline{\Omega}_{\mathcal{A}_{(zxyxtz)^\omega xyx}} \mathbf{Sd} \rightarrow \overline{\Omega}_X \mathbf{S}$ such that $\varepsilon(l) = l$, for every $l \in \{x, y, z, t\}$, is compatible with $\mathbf{V}(B_2)$, since $B_2 \models \pi z = \pi xy = \pi xt = \pi$. Hence, by theorem 2.1,

$$\mathbf{V}^{i+1}(B_2) \models (zxyxtz)^\omega x(yx)^2 = (zxyxtz)^\omega xyx.$$

So, for every natural n , $\mathbf{V}^n(B_2) \models (zxyxtz)^\omega x(yx)^2 = (zxyxtz)^\omega xyx$ and, consequently,

$$\mathbf{V}^*(B_2) \models (zxyxtz)^\omega x(yx)^2 = (zxyxtz)^\omega xyx.$$

4. DECIDABILITY OF $\mathbf{V}^*(B_2)$

A pseudovariety \mathbf{V} is decidable if there is an algorithm to test whether a finite semigroup is or is not in \mathbf{V} . Theorem 3.8 does not lead to an effective construction of a basis of pseudoidentities for the semidirect powers of $\mathbf{V}(B_2)$, but it nevertheless suggests a proof of decidability of $\mathbf{V}^*(B_2)$.

Theorem 4.1. *Let S be a finite semigroup with m elements. Then, $S \in \mathbf{V}^*(B_2)$ if and only if $S \in \mathbf{V}^{m+2}(B_2)$.*

Proof

To prove the non trivial implication, let $S \in \mathbf{V}^*(B_2)$. Then there is $k \geq 1$ such that $S \in \mathbf{V}^k(B_2)$. If $k \leq m+2$, then $\mathbf{V}^k(B_2) \subseteq \mathbf{V}^{m+2}(B_2)$ and, consequently, $S \in \mathbf{V}^{m+2}(B_2)$.

Suppose $k > m+2$. Proving that $S \in \mathbf{V}^{m+2}(B_2)$ is equivalent to proving that the pseudoidentities of a basis of $\mathbf{V}^{m+2}(B_2)$ hold in S . By theorem 3.8, we may take all pseudoidentities of the form:

$$\pi_{m+1}\varepsilon_{m+1}(\pi_m \cdots \varepsilon_2(\pi_1\varepsilon_1(u)) \dots) = \pi_{m+1}\varepsilon_{m+1}(\pi_m \cdots \varepsilon_2(\pi_1\varepsilon_1(v)) \dots)$$

where $u, v, \pi_i, \varepsilon_i$ are in the conditions described in the theorem 3.8, for $i = 1, 2, \dots, m+1$. Let λ denote one of these pseudoidentities and, for $m+1 \geq i \geq j \geq 1$,

$$\begin{aligned} p_{i,j} &= \varepsilon_{m+1} \circ \cdots \circ \varepsilon_{i+1}(\pi_i \varepsilon_i(\pi_{i-1} \cdots \varepsilon_{j+1}(\pi_j) \dots)) \\ p_{i,0}^u &= \varepsilon_{m+1} \circ \cdots \circ \varepsilon_{i+1}(\pi_i \varepsilon_i(\pi_{i-1} \cdots \varepsilon_1(u) \dots)) \\ p_{i,0}^v &= \varepsilon_{m+1} \circ \cdots \circ \varepsilon_{i+1}(\pi_i \varepsilon_i(\pi_{i-1} \cdots \varepsilon_1(v) \dots)) \\ q_{i,j} &= \pi_i \varepsilon_i(\pi_{i-1} \cdots \varepsilon_{j+1}(\pi_j) \dots) \\ q_{i,0}^u &= \pi_i \varepsilon_i(\pi_{i-1} \cdots \varepsilon_1(u) \dots) \\ q_{i,0}^v &= \pi_i \varepsilon_i(\pi_{i-1} \cdots \varepsilon_1(v) \dots) \end{aligned}$$

Note that λ is the pseudoidentity $q_{m+1,0}^u = q_{m+1,0}^v$.

The expressions

$$\begin{aligned} p_{1,1} &= \varepsilon_{m+1} \circ \cdots \circ \varepsilon_2(\pi_1) \\ p_{2,1} &= \varepsilon_{m+1} \circ \cdots \circ \varepsilon_3(\pi_2 \varepsilon_2(\pi_1)) \\ &\vdots \\ p_{m,1} &= \varepsilon_{m+1}(\pi_m \varepsilon_m(\cdots \varepsilon_2(\pi_1) \dots)) \\ q_{m+1,1} &= \pi_{m+1} \varepsilon_{m+1}(\pi_m \varepsilon_m(\cdots \varepsilon_2(\pi_1) \dots)) \end{aligned}$$

are factors of both members of λ and represent $m+1$ elements of $\overline{\Omega}_{X_{m+1}}\mathbf{S}$. Hence, for each map $\varphi : X_{m+1} \longrightarrow S$, at least two of these expressions have the same image under the mapping $\overline{\varphi} : \overline{\Omega}_{X_{m+1}}\mathbf{S} \longrightarrow S$. For a fixed φ , suppose that these two expressions have indices i and j and that $i > j$. In case $m+1 > i > j \geq 1$, then $\overline{\varphi}(p_{i,1}) = \overline{\varphi}(p_{j,1})$, otherwise, $m+1 = i > j \geq 1$ and $\overline{\varphi}(q_{m+1,1}) = \overline{\varphi}(p_{j,1})$. Consider the case where $m+1 > i > j \geq 1$ (the proof is similar if $m+1 = i > j \geq 1$). Then, for $l \geq 1$, $\overline{\varphi}(p_{i,j+1})^l \cdot \overline{\varphi}(p_{j,1}) = \overline{\varphi}(p_{j,1})$. Consequently, applying $\overline{\varphi}$ to both members of λ , we conclude that

$$\begin{aligned} \overline{\varphi}(q_{m+1,0}^u) &= \overline{\varphi}(q_{m+1,0}^v) \\ (2) \quad &\text{if and only if} \\ \overline{\varphi}(q_{m+1,i+1}(p_{i,j+1})^l p_{j,0}^u) &= \overline{\varphi}(q_{m+1,i+1}(p_{i,j+1})^l p_{j,0}^v). \end{aligned}$$

Since λ belongs to a basis of pseudoidentities of $\mathbf{V}^{m+2}(B_2)$, then the pseudoidentity

$$\varrho_1 = (q_{i,0}^u = q_{i,0}^v)$$

holds in $\mathbf{V}^{i+1}(B_2)$ and, by theorem 2.3, $g\mathbf{V}^{i+1}(B_2) \models (\varrho_1, \mathcal{A}_{\pi_i})$. Let

$$(3) \quad \begin{aligned} X'_{i+1} &= X_i, \\ \varepsilon'_{i+1} : \quad \overline{\Omega}_{\mathcal{A}_{\pi_i}} \mathbf{Sd} &\longrightarrow \overline{\Omega}_{X'_{i+1}} \mathbf{S} \\ x \in E(\mathcal{A}_{\pi_i}) &\longmapsto x, \\ \pi'_{i+1} &= q_{i,j+1}. \end{aligned}$$

The labelings of \mathcal{A}_{π_i} defined by π'_{i+1} and ε'_{i+1} are compatible with $\mathbf{V}(B_2)$ if and only if

$$B_2 \models \pi'_{i+1} \varepsilon'_{i+1}(w_q) = \pi'_{i+1} \varepsilon'_{i+1}(w'_q),$$

for $q \in V(\mathcal{A}_{\pi_i})$ and $w_q, w'_q \in \mathcal{A}_{\pi_i}^*(i(\pi_i)\delta_{\pi_i}, q)$, which we may rewrite as $B_2 \models q_{i,j+1}w_q = q_{i,j+1}w'_q$. Note that $q_{i,j+1}, w_q, w'_q \in E(\overline{\Omega}_{\mathcal{A}_{\pi_i}} \mathbf{Sd})$ and $i(w_q)\delta_{\pi_i}i(w'_q)$ so, the set of pairs

$$\{(t(q_{i,j+1})^{-1}, i(w_q))\} \cup \delta_{\pi_i}$$

generate the same equivalence relation as the set

$$\{(t(q_{i,j+1})^{-1}, i(w'_q))\} \cup \delta_{\pi_i}.$$

Hence, the compatibility conditions are satisfied and theorem 2.1 leads to the conclusion that the pseudoidentity

$$\varrho_2 = \left((q_{i,j+1})^2 \cdot \varepsilon_i \circ \dots \circ \varepsilon_{j+1}(q_{j,0}^u) = (q_{i,j+1})^2 \cdot \varepsilon_i \circ \dots \circ \varepsilon_{j+1}(q_{j,0}^v) \right)$$

holds in $\mathbf{V}^{i+2}(B_2)$, and, by theorem 2.3, $g\mathbf{V}^{i+2}(B_2) \models (\varrho_2, \mathcal{A}_{\pi'_{i+1}})$ (note that $\mathcal{A}_{\pi'_{i+1}} = \mathcal{A}_{\pi_i}$).

Using the same arguments, by theorems 2.3 and 2.1, making $k - (m + 2)$ times the choice of the labelings as in (3), we conclude that the following pseudoidentity, denoted $\varrho_{k-(m+1)}$,

$$(q_{i,j+1})^{k-(m+1)} \cdot \varepsilon_i \circ \dots \circ \varepsilon_{j+1}(q_{j,0}^u) = (q_{i,j+1})^{k-(m+1)} \cdot \varepsilon_i \circ \dots \circ \varepsilon_{j+1}(q_{j,0}^v)$$

holds in $\mathbf{V}^{i+k-(m+1)}$. Consequently,

$$g\mathbf{V}^{i+k-(m+1)}(B_2) \models (\varrho_{k-(m+1)}, \mathcal{A}_{\pi_i}).$$

Now, for $t = 1, \dots, (m + 1 - i)$, making the choices:

$$\begin{aligned} X'_{i+k-(m+1)+t} &= X_{i+t} \\ \pi'_{i+k-(m+1)+t} &= \pi_{i+t} \\ \varepsilon'_{i+k-(m+1)+t} &= \varepsilon_{i+t} \end{aligned}$$

we obtain labelings of $\mathcal{A}_{\pi'_{i+k-(m+2)+t-1}} (= \mathcal{A}_{\pi_{i+t-1}})$ compatible with $\mathbf{V}(B_2)$, since the pseudoidentity λ holds in $\mathbf{V}^{m+2}(B_2)$. Hence

$$q_{m+1,i+1} \cdot (p_{i,j+1})^{k-(m+1)} \cdot p_{j,0}^u = q_{m+1,i+1} \cdot (p_{i,j+1})^{k-(m+1)} \cdot p_{j,0}^v$$

holds in $\mathbf{V}^k(B_2)$. Since $S \in \mathbf{V}^k(B_2)$, this pseudoidentity holds in S and the equivalence (2) leads to the conclusion that $\overline{\varphi}(q_{m+1,0}^u) = \overline{\varphi}(q_{m+2,0}^v)$. So, S satisfies λ .

Finally we can say that $S \in \mathbf{V}^{m+2}(B_2)$, because S satisfies all pseudoidentities of the basis of $\mathbf{V}^{m+2}(B_2)$. \square

Theorem 4.2. *The pseudovariety of semigroups $\mathbf{V}^*(B_2)$ is decidable.*

Proof

The pseudovariety $\mathbf{V}(B_2)$ is decidable, because it is finitely generated [1]. In fact, for X a finite set, $\overline{\Omega}_X \mathbf{V}(B_2)$ is finite and computable (these two conditions are equivalent to saying that $\mathbf{V}(B_2)$ is order-computable, property that implies decidability [2]).

By [1, theorem 10.2.3], there is a continuous embedding from $\overline{\Omega}_X \mathbf{V}^{n+1}(B_2)$ into $\overline{\Omega}_Y \mathbf{V}^n(B_2) * \Omega_X \mathbf{V}(B_2)$, for every $n \geq 1$, where $Y = X \times (\Omega_X \mathbf{V}(B_2))^1$. Hence, if $\overline{\Omega}_Y \mathbf{V}^n(B_2)$ and $\Omega_X \mathbf{V}(B_2)$ are finite and computable then $\overline{\Omega}_X \mathbf{V}^{n+1}(B_2)$ is finite and computable.

By induction on n , we conclude that for every $n \geq 1$, the free finitely generated semigroups of $\mathbf{V}^n(B_2)$ are finite and computable. Hence, $\mathbf{V}^n(B_2)$ is decidable, for every $n \geq 1$.

Given a semigroup S , by theorem 4.1, $S \in \mathbf{V}^*(B_2)$ if and only if $S \in \mathbf{V}^m(B_2)$, where $m = \sharp(S) + 2$, and, since $\mathbf{V}^m(B_2)$ is decidable, we conclude that it is decidable if S is or is not an element of $\mathbf{V}^*(B_2)$. \square

5. ANOTHER ALGORITHM TO SOLVE THE MEMBERSHIP PROBLEM FOR $\mathbf{V}^*(B_2)$

In order to establish effective conditions to verify the membership relation to $\mathbf{V}^*(B_2)$, as an alternative to theorem 4.1, we follow the suggestions given by J. Almeida in [1, section 10.10] to obtain a proof of decidability of $\mathbf{V}^*(B_2)$. Given a semigroup $S \in \mathbf{ER} \cap \mathbf{LR}$ a decomposition of S based on a theorem of P. Stiffler [12] was suggested, and the problem becomes to test membership in $\mathbf{V}^*(B_2)$ of some factors of S .

Given a finite semigroup S , $Id_m(S)$ represents the set of its $[0]$ -minimal ideals and, for $s \in S$, J_s represents the equivalence class of s

for the Green relation \mathcal{J} . So S is a subdirect product of the semigroups

$$S/\{s \in S : J_s \not\supseteq I_j \setminus \{0\}, I_j \in Id_m(S)\}$$

which are homomorphic images of S and have a unique $[0]$ -minimal ideal. Hence, $S \in \mathbf{V}$ if and only if all these factors belong to \mathbf{V} . Therefore, in the study of the membership relation to a pseudovariety, we may assume that S has a unique $[0]$ -minimal ideal, I . The semigroup I is completely $[0]$ -simple or nilpotent and $I \cong \mathcal{M}^{[0]}(G, R, C, P)$, a Rees matrix semigroup, where P is a regular matrix or a null matrix, respectively. The set C may be identified with the set of \mathcal{L} -classes (equivalence classes for the Green relation \mathcal{L}) of I , which do not contain the zero element, if it exists. We represent by C^* the set of all \mathcal{L} -classes of I . Consider the right action of S on C^* defined by the following morphism from S to the semigroup \mathcal{T}_{C^*} of all functions $C^* \rightarrow C^*$

$$\begin{array}{ccc} \vartheta_S : S & \longrightarrow & \mathcal{T}_{C^*} \\ s & \longmapsto & t_s \end{array} \quad \text{where} \quad \begin{array}{ccc} t_s : C^* & \longrightarrow & C^* \\ L & \longmapsto & Ls = \{ls : l \in L\} . \end{array}$$

The subsemigroup $\vartheta_S(S)$ of \mathcal{T}_{C^*} is represented by $RLM(S)$.

The following theorem has been proved by P. Stiffler and establishes a decomposition of a semigroup in terms of semidirect product. For S and T semigroups, by $S \odot T$ we denote the wreath product of S by T which is the semidirect product of $S^{(T^1)}$ by T .

Theorem 5.1. [12, theorem 3.1] *With the above notation, let S be a finite semigroup with a unique $[0]$ -minimal ideal I such that $I \cong \mathcal{M}^{[0]}(G, R, C, P)$. Then,*

- i. *in case I is regular or $\sharp C > 1$,*
 - (a) *if $S \neq I$, $S \prec G \odot M \odot RLM(S) \odot S/I$,*
 - (b) *if $S = I$, $S \prec G \odot M \odot RLM(S) \odot U$;*
- ii. *in case $\sharp C = 1$,*
 - (a) *if $S \neq I$, $S \prec G \odot M \odot Sl_2 \odot S/I$,*
 - (b) *if $S = I$, $S \prec G \odot M \odot Sl_2 \odot U$;*

where M is any monoid such that $\sharp M \geq \sharp R$, and U is any of the semigroups $B(1, 2)$, the right-zero semigroup with two elements, and Sl_2 , the semilattice with two elements. \square

If S is aperiodic then the group G is trivial and if we choose:

- $U = Sl_2$;
- $M = \langle 1, x_1, \dots, x_r \mid x^2 = x, xy = yx, 1x = x, x, y \in \{1, \dots, x_r\} \rangle$
where $r = \sharp R$,

then the factors of S in Stiffler's decomposition are the semigroups $RLM(S)$ and S/I , and elements of \mathbf{Sl} , the pseudovariety of all semi-lattices. Consequently, if \mathbf{V} is a pseudovariety of aperiodic semigroups such that $\mathbf{Sl} \subseteq \mathbf{V}$, then, $S \in \mathbf{V}^*$ if and only if S/I and $RLM(S)$ are elements of \mathbf{V}^* .

The arguments we are going to develop involve induction on $\#S$. Note that, for a finite non trivial semigroup S with a unique $[0]$ -minimal I , $\#(S/I) < \#S$ and $\#RLM(S) \leq \#S$. If $\#RLM(S) = \#S$, then $RLM(S) \cong S$. So, the process of successive decompositions is finite and each terminal semigroup T satisfies, at least, one of the following conditions: T is trivial, T satisfies the conditions of theorem 5.1 case (iib), or $T \cong RLM(T)$.

The structure of the semigroups T such that $T \cong RLM(T)$ was studied in [1, section 10.10] and the result is presented in the next proposition.

Proposition 5.2. *Let $S \in \mathbf{ER} \cap \mathbf{LR}$ with a unique $[0]$ -minimal ideal I . With the above notation, if $S \cong RLM(S)$ then $I \cong B_{\#C}$ and S/I is nilpotent. \square*

Results of section 3 allow us to improve proposition 5.2.

Proposition 5.3. *Let $S \in \mathbf{ER} \cap \mathbf{LR}$ be a semigroup with a unique $[0]$ -minimal ideal I and $S \cong RLM(S)$. Then $S \in \mathbf{V}^*(B_2)$ if and only if $S = I$.*

Proof

If $S = I$ then, by proposition 5.2, $S \cong B_n$ where $n+1$ is the number of \mathcal{L} -classes of I . Since $B_n \in \mathbf{V}(B_2)$, then $S \in \mathbf{V}^*(B_2)$.

If $S \neq I$ then there is $\sigma \in S \setminus I$. Each $s \in S$ induces a partial transformation on C , by restriction of t_s to C , that is injective. Let us represent the corresponding morphism as $\vartheta'_S : S \rightarrow \mathcal{I}_C$, where \mathcal{I}_C is the semigroup of partial injective transformations on C . Since ϑ_S is injective and 0 is a fixed point for transformations of $RLM(S)$, $S \cong RLM(S) \cong \vartheta'_S(S)$. For $i = 1, \dots, n$, let L_i be the \mathcal{L} -classes of I which are elements of C . The non null elements of $\vartheta'_S(I)$ are the partial injective transformations, represented by $s_{p,q}$, whose domain is a singular set, $\{L_p\}$, and whose image set is $\{L_q\}$, respectively, where $p, q \in \{1, \dots, n\}$. By 0 we represent the empty transformation. So the domain of the partial injective transformation $\sigma' = \vartheta'_S(\sigma)$ has two or more elements, which means that σ' transforms L_i in L_j and L_k in L_l , for some $i, j, k, l \in \{1, \dots, n\}$ such that $i \neq k$.

Consider the pseudoidentity

$$(zxyxtz)^\omega x(yx)^2 = (zxyxtz)^\omega xyx$$

which holds in $\mathbf{V}^*(B_2)$ (as has been proved at the end of section 3) and the morphism $\bar{\psi} : \bar{\Omega}_{\{x,y,z,t\}} \mathbf{S} \rightarrow \vartheta'_S(S)$, continuous extension of $\psi : \{x, y, z, t\} \rightarrow \vartheta'_S(S)$ which is defined by

$$\psi(x) = \sigma', \quad \psi(y) = s_{j,k}, \quad \psi(z) = s_{i,i}, \quad \psi(t) = s_{l,i}.$$

$$\begin{aligned} \text{Consequently, } \bar{\psi}((zxyxtz)^\omega) &= s_{i,i}, \\ \bar{\psi}(x(yx)^2) &= 0, \\ \bar{\psi}(xyx) &= s_{i,l}, \end{aligned}$$

and $\bar{\psi}((zxyxtz)^\omega x(yx)^2) \neq \bar{\psi}((zxyxtz)^\omega xyx)$, which means that $S \notin \mathbf{V}^*(B_2)$. \square

Consequently, we obtain another proof that $\mathbf{V}^*(B_2) \neq \mathbf{ER} \cap \mathbf{LR}$ (see [13]) since there are semigroups S in the conditions of the last proposition and such that $S \neq I$. As an example, consider S the subsemigroup of $\mathcal{T}_{\{0,1,2,3,4,5\}}$ generated by the transformations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 4 & 1 \end{pmatrix}.$$

Now we can write an algorithm to test if a member of $\mathbf{ER} \cap \mathbf{LR}$ belongs to $\mathbf{V}^*(B_2)$:

Algorithm 1

ENTRY: $S \in \mathbf{ER} \cap \mathbf{LR}$

- 1.- If $\sharp S = 1$ then $S \in \mathbf{V}^*(B_2)$.
- 2.- If $\sharp S > 1$ then:
 - 2.1- let $F = \{I_j : I_j \in Id_m(S)\}$;
 - 2.2- for each $I_j \in F$:
 - 2.2.1- $S_j = S / \{s \in S : J_s \not\supseteq I_j \setminus \{0\}\}$
 - 2.2.2- if $RLM(S_j) \cong S_j$ and $S_j = I_j$ then $S_j \in \mathbf{V}^*(B_2)$;
 - 2.2.3- if $RLM(S_j) \cong S_j$ and $S_j \neq I_j$ then $S_j \notin \mathbf{V}^*(B_2)$;
 - 2.2.4- if $RLM(S_j) \not\cong S_j$ then,
 - if $RLM(S_j), S_j/I_j \in \mathbf{V}^*(B_2)$
 - then $S_j \in \mathbf{V}^*(B_2)$;
 - else $S_j \notin \mathbf{V}^*(B_2)$;
 - 2.3- if, for $j = 1, \dots, \sharp F$, $S_j \in \mathbf{V}^*(B_2)$ then $S \in \mathbf{V}^*(B_2)$ else $S \notin \mathbf{V}^*(B_2)$.

END

Further considerations on the factors of S in Stiffler's decomposition are stated in the following results, which lead to an improved algorithm.

Proposition 5.4. *If $S \in \mathbf{ER} \cap \mathbf{LR}$ is a regular semigroup then $S \in \mathbf{V}^*(B_2)$.*

Proof

The factors of S in the subdirect product, that is the semigroups

$$S_j = S / \{s \in S : J_s \not\geq I_j \setminus \{0\}, I_j \in Id_m(S)\}$$

are regular and each has only one $[0]$ -minimal ideal. By proposition 5.2, each one of those semigroups S_j verifies either $S_j \not\cong RLM(S_j)$ or S_j is equal to its 0-minimal ideal and so $S_j \in \mathbf{V}^*(B_2)$.

Suppose $S_j \not\cong RLM(S_j)$, for some index j . The semigroups S_j/I_j and $RLM(S_j)$ are again regular semigroups of $\mathbf{ER} \cap \mathbf{LR}$ and consequently, for each one, we may initialize a new decomposition.

Since S is finite, this process finishes after a finite number of steps, and the terminal semigroups belong to $\mathbf{V}^*(B_2)$, because they are trivial semigroups, or satisfy conditions of theorem 5.1 case (iib), or are equal to the 0-minimal ideal which is of the form B_n , by proposition 5.2. The final conclusion is that $S \in \mathbf{V}^*(B_2)$. \square

Corollary 5.5. *Let $S \in \mathbf{ER} \cap \mathbf{LR}$ be a semigroup equal to its unique $[0]$ -minimal ideal. Then $S \in \mathbf{V}^*(B_2)$.*

Proof

Under the hypotheses, S is a completely $[0]$ -simple semigroup, and so is regular, or S is isomorphic to a nilpotent semigroup with two elements. So, in both cases, $S \in \mathbf{V}^*(B_2)$. \square

Lemma 5.6. *Let $S \in \mathbf{ER}$ be a semigroup with only one $[0]$ -minimal ideal I , which is regular. Then the semigroup $RLM(S)$ has only one $[0]$ -minimal ideal, which is an aperiodic Brandt semigroup.*

Proof

The following proof contains arguments similar to the arguments used by J. Almeida to obtain the result stated in lemma 5.2.

If S does not have a zero and since $I \in \mathbf{ER}$, then I has only one \mathcal{L} -class and $RLM(S)$ is trivial. So, suppose S is a not trivial semigroup which has a zero. Hence, $RLM(S)$ is non trivial and is the homomorphic image of S under ϑ_S . Consequently, $\vartheta_S(I)$ is regular and is the unique 0-minimal ideal of $RLM(S)$.

Let $e, f \in I$ be two idempotents. If $e\mathcal{L}f$ then $\vartheta_S(e) = \vartheta_S(f)$, else $\vartheta_S(e)$ and $\vartheta_S(f)$ are not \mathcal{L} -equivalent. So, $\vartheta_S(I)$ does not contain

\mathcal{L} -equivalent idempotents and, since $\vartheta_S(I) \in \mathbf{ER}$, does not contain \mathcal{R} -equivalent idempotents. Consequently, the product of idempotents in $\vartheta_S(I)$ is commutative. Hence we conclude that $\vartheta_S(I)$ is a completely 0-simple inverse semigroup, which means that it is a Brandt semigroup.

Moreover, since I does not contain \mathcal{R} -equivalent idempotents, the elements of $\vartheta_S(I)$ are transformations such that only one element of the domain may have image not zero. So $\vartheta_S(I)$ is aperiodic \square

The semigroup $RLM(S)/\vartheta_S(I)$ is the image of S/I by a morphism ϑ such that $p_{\vartheta_S(I)} \circ \vartheta_S = \vartheta \circ p_I$, where $p_{\vartheta_S(I)}$ is the projection of $RLM(S)$ in $RLM(S)/\vartheta_S(I)$ and p_I is the projection of S in S/I . If S satisfies lemma 5.6's conditions, then $RLM(S)$ satisfies them too, and so $RLM(RLM(S))$ has only one 0-minimal ideal, which is a Brandt semigroup isomorphic to the 0-minimal ideal of $RLM(S)$, as we can deduce from the proof of lemma 5.6. Consequently, $RLM(RLM(RLM(S)))$ is isomorphic to $RLM(RLM(S))$.

If S has only one $[0]$ -minimal ideal I which is non regular, then I is nilpotent, $\vartheta_S(I) = 0$ and $RLM(S)$ is an image of S/I . In this case, $RLM(S)$ can have several 0-minimal ideals and $\sharp RLM(S) < \sharp S$.

Given $S \in \mathbf{ER} \cap \mathbf{LR}$ with r \mathcal{J} -classes that do not contain 0, $(J_i)_{i \in \{1, \dots, r\}}$, to execute the algorithm 1 is equivalent to test if each semigroup of the family

$$\mathcal{F} = (S/\{a \in S : J_a \not\leq J_i\})_{i \in \{1, \dots, r\}}$$

is an element of $\mathbf{V}^*(B_2)$ and, $S \in \mathbf{V}^*(B_2)$ if and only if $\mathbf{V}^*(B_2)$ contains \mathcal{F} . Note that \mathcal{F} is partially ordered by the order relation \prec_q defined by: $T \prec_q D$ if and only if T is a quotient of D . The minimum element of \mathcal{F} is the trivial semigroup and there are as many maximal elements as 0-minimal ideals. If a semigroup covers the trivial semigroup then it is a completely $[0]$ -simple semigroup or a nilpotent semigroup with two elements.

For each $S_i \in \mathcal{F}$, we represent its 0-minimal ideal by I_i . So, S_i/I_i is a subdirect product of $\mathcal{F}_i = (S_{i_k})_{i_k \in K}$, where $K \subset \{1, \dots, r\}$ and each $S_{i_k} \in \mathcal{F}_i$ satisfies $S_{i_k} \prec_q S_i$ and $S_{i_k} \neq S_i$. Hence, the criterion to test if $S_i \in \mathbf{V}^*(B_2)$ is the following:

1. if $S_i = I_i$ or S_i is regular then $S_i \in \mathbf{V}^*(B_2)$;
2. else, if $S_i \cong RLM(S_i)$ then $S_i \notin \mathbf{V}^*(B_2)$;
3. otherwise, $S \in \mathbf{V}^*(B_2)$ if and only if $\mathcal{F}_i \cup \{RLM(S_i)\} \subseteq \mathbf{V}^*(B_2)$.

In case 3 and if I_i is not regular then $RLM(S_i)$ is a morphic image of S_i/I_i , then $RLM(S_i) \in \mathbf{V}^*(B_2)$ if and only if the semigroups $S_j \in \mathcal{F}$, such that $S_j \prec_q (S_i/I_i)$, are members of $\mathbf{V}^*(B_2)$. If I_i is regular then,

by proposition 5.6, $RLM(S_i)$ has only one 0-minimal ideal, which is a Brandt semigroup and, $RLM(S_i) \in \mathbf{V}^*(B_2)$ if and only if the semigroups $S_j \in \mathcal{F}$, such that $S_j \prec_q (S_i/I_i)$, are members of $\mathbf{V}^*(B_2)$ and $RLM(RLM(S_i))$ is completely 0-simple.

So we can write a new algorithm.

Algorithm 2

ENTRY: $S \in \mathbf{ER} \cap \mathbf{LR}$

- 1.- If either S is regular or S is nilpotent then $S \in \mathbf{V}^*(B_2)$.
- 2.- Else :
 - 2.1- $J = \{J_j : J_j \text{ non null } \mathcal{J}\text{-class of } S\}$;
 - 2.2- for each $j \in \{1, \dots, \#J\}$
 - 2.2.1- let $S_j = S/\{s \in S : J_s \not\subseteq J_j\}$;
 - 2.2.2- if either S_j is regular or, $\#S_j = 2$ and S_j is nilpotent,
then $S_j \in \mathbf{V}^*(B_2)$;
 - 2.2.3- else if $RLM(S_j) \cong S_j$ then $S_j \notin \mathbf{V}^*(B_2)$;
 - 2.2.4- else if $RLM(RLM(S_j))$ is equal to its 0-minimal ideal
then $S_j \in \mathbf{V}^*(B_2)$;
else $S_j \notin \mathbf{V}^*(B_2)$;
 - 2.3- if, for some $1 \leq j \leq \#J$, $S_j \notin \mathbf{V}^*(B_2)$
then $S \notin \mathbf{V}^*(B_2)$;
else $S \in \mathbf{V}^*(B_2)$.

END

Let S be a finite semigroup with n elements. Testing if $S \in \mathbf{ER} \cap \mathbf{LR}$ is equivalent to testing if $S \models (exeye)^\omega x = (exeye)^\omega$, and this can be done by an algorithm of $O(n^3)$ using a table for the function that associates s^ω to each $s \in S$, which can be computed in polynomial time of $O(n^2)$. The execution of algorithm 2 is the result of the execution of a finite number of basic steps, which is bounded by a linear function on $\#J$. The basic steps are to test if a semigroup is regular or nilpotent, to compute the \mathcal{J} -classes of S , the Rees-quotients of the form S_j and their homomorphic images $RLM(S_j)$ and $RLM(RLM(S_j))$, and to test if $RLM(S_j) \cong S_j$. To verify if S is regular or nilpotent we need to compute no more than $3n^2$ products of elements of S . The calculation of the Green classes structure of S takes $2n^2$ products, since Green classes of S are defined by comparison of sets of the form aS^1 and of sets of the form S^1a , for all $a \in S$. If we know the Green classes structure of S , the computation of semigroups S_j and of their Green classes structure is trivial. Given a semigroup T , if we know its Green classes structure, to compute $RLM(T)$ takes less than $\#T^2$ products.

So, the computation of the semigroups $RLM(S_j)$ and $RLM(RLM(S_j))$ takes less than $2n^2$ products. Testing if $RLM(S_j) \cong S_j$ is equivalent to testing if $\sharp RLM(S_j) = \sharp S_j$. Consequently, we have an algorithm of polynomial complexity of $O(n^3)$ to test if a finite semigroup S with n elements belongs to $\mathbf{V}^*(B_2)$.

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